

Distance preserving mappings from ternary vectors to permutations

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Abstract

Distance-preserving mappings (DPMs) are mappings from the set of all q -ary vectors of a fixed length to the set of permutations of the same or longer length such that every two distinct vectors are mapped to permutations with the same or even larger Hamming distance than that of the vectors. In this paper, we propose a construction of DPMs from ternary vectors. The constructed DPMs improve the lower bounds on the maximal size of permutation arrays.

Key words: distance-preserving mappings, distance-increasing mappings, permutation arrays, Hamming distance

1 Introduction

A mapping from the set of all q -ary vectors of length m to the set of all permutations of $\{1, 2, \dots, n\}$ is called a distance-preserving mapping (DPM) if every two distinct vectors are mapped to permutations with the same or even larger Hamming distance mutual than that of the vectors. A distance-increasing mapping (DIM) is a special DPM such that the distances are strictly increased except when that is obviously not possible. DPMs and

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DIMs are useful for the construction of permutation arrays (PAs) which are applied to various applications, such as trellis code modulations and power line communications [7], [8], [9], [10], [11], [13], [21], [22], [23], [24]. All DPMs and DIMs proposed so far are from binary vectors: [2], [3], [4], [5], [6], [12], [14], [15], [16], [17], [19], [20]. In this paper we propose a general construction method to construct DPMs or DIMs from ternary vectors. By using this method, we construct DIMs for $n = m + 2$ for $m \geq 3$, DPMs for $n = m + 1$ for $m \geq 9$, and DPMs for $n = m$ for $m \geq 13$.

The paper is organized as follows. In the next section we introduce some notations and state our main results. In Section 3 we introduce a general recursive construction of DPMs and DIMs. In Sections 4 and 5 we introduce mappings that can be used to start the recursion in the three cases we consider. Finally, in an appendix, we give explicit listings of the values of some mappings that are used as building blocks to construct the mappings given in Sections 4 and 5.

2 Notations and main results

Let S_n denote the set of all $n!$ permutations of $F_n = \{1, 2, \dots, n\}$. A permutation $\pi : F_n \rightarrow F_n$ is represented by an n -tuple $\pi = (\pi_1, \pi_2, \dots, \pi_n)$ where $\pi_i = \pi(i)$. Let Z_3^n denote the set of all ternary vectors of length n . The Hamming distance between two n -tuples $\mathbf{a} = (a_1, a_2, \dots, a_n)$ and $\mathbf{b} = (b_1, b_2, \dots, b_n)$ is denoted by $d_H(\mathbf{a}, \mathbf{b})$ and is defined as

$$d_H(\mathbf{a}, \mathbf{b}) = |\{j \in F_n : a_j \neq b_j\}|.$$

Let $\mathcal{F}_{n,k}$ be the set of injective functions from Z_3^n to S_{n+k} . Note that $\mathcal{F}_{n,k}$ is empty if $(n+k)! < 3^n$.

For $k \geq 0$, let $\mathcal{P}_{n,k}$ be the set of functions in $\mathcal{F}_{n,k}$ such that

$$d_H(f(\mathbf{x}), f(\mathbf{y})) \geq d_H(\mathbf{x}, \mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in Z_3^n$. These mappings are called distance preserving mappings (DPM).

For $k \geq 1$, let $\mathcal{I}_{n,k}$ be the set of functions in $\mathcal{F}_{n,k}$ such that

$$d_H(f(\mathbf{x}), f(\mathbf{y})) > d_H(\mathbf{x}, \mathbf{y}) \tag{1}$$

for all distinct $\mathbf{x}, \mathbf{y} \in Z_3^n$. These mappings are called distance increasing mappings (DIM).

Our main result is the following theorem.

Theorem 1 a) $\mathcal{I}_{n,2}$ is non-empty for $n \geq 3$.

b) $\mathcal{P}_{n,1}$ is non-empty for $n \geq 9$.

c) $\mathcal{P}_{n,0}$ is non-empty for $n \geq 13$.

The proof of the theorem is constructive. A relatively simple recursive method is given (in the next section) to construct a mapping of length $n + 1$ from a mapping of length n . Explicit mappings that start the recursion in the three cases are given in last part of the paper, including the appendix.

An (n, d) permutation array (PA) is a subset of S_n such that the Hamming distance between any two distinct permutations in the array is at least d . An $(n, d; q)$ code is a subset of vectors (codewords) of length n over an alphabet of size q and with distance at least d between distinct codewords. One construction method of PAs is to construct an (n, d') -PA from an $(m, d; q)$ code using DPMs or DIMs. More precisely, if C is an $(m, d; q)$ code and there exists a DPM f from Z_q^m to S_n , then $f(C)$ is an (n, d) PA. If f is DIM, then $f(C)$ is an $(n, d+1)$ PA. This has been a main motivation for studying DPMs. Let $P(n, d)$ denote the largest possible size of an (n, d) -PA. The exact value of $P(n, d)$ is still an open problem in most cases, but we can lower bound this value by the maximal size of a suitable code provided a DPM (or DIM) is known. Let $A_q(n, d)$ denote the largest possible size of an (n, d) code over a code alphabet of size q . In [5], Chang et al. used this approach to show that for $n \geq 4$ and $2 \leq d \leq n$, we have $P(n, d) \geq A_2(n, d - 1)$. In [17], Chang further improved the bound to $P(n, d) \geq A_2(n, d - \delta)$ for $n \geq n_\delta$ and $\delta + 1 \leq d \leq n$ where $\delta \geq 2$ and n_δ is a positive integer determined by δ , e.g. $n_2 = 16$.

From Theorem 1 we get the following bounds.

Theorem 2 a) For $n \geq 5$ and $2 \leq d \leq n$, we have

$$P(n, d) \geq A_3(n - 2, d - 1).$$

b) For $n \geq 10$ and $2 \leq d \leq n$, we have

$$P(n, d) \geq A_3(n - 1, d).$$

c) For $n \geq 13$ and $2 \leq d \leq n$, we have

$$P(n, d) \geq A_3(n, d).$$

Bounds on $A_2(n, d)$ and $A_3(n, d)$ have been studied by many researchers, see e.g. [18, Ch.5] and [1]. In general, the lower bounds on $P(n, d)$ obtained from use of ternary codes are better than those obtained from binary codes. For example, using Chang's bound [17], we get $P(16, 5) \geq A_2(16, 3) \geq 2720$, whereas Theorem 2 gives $P(16, 5) \geq A_3(16, 5) \geq 19683$. Similarly, we get $P(16, 9) \geq A_2(16, 7) \geq 36$ and $P(16, 9) \geq A_3(16, 9) \geq 243$.

3 The general recursive construction.

For any array $\mathbf{u} = (u_1, u_2, \dots, u_n)$, we use the notation $\mathbf{u}_i = u_i$.

We start with a recursive definition of functions from Z_3^n to S_{n+k} . For $f \in \mathcal{F}_{n,k}$, define $g = H(f) \in \mathcal{F}_{n+1,k}$ as follows. Let $\mathbf{x} = (x_1, x_2, \dots, x_n) \in Z_3^n$ and $f(\mathbf{x}) = (\varphi_1, \varphi_2, \dots, \varphi_{n+k})$. Suppose that the element $n+k-4$ occurs in position r , that is $\varphi_r = n+k-4$. Then

$$\begin{aligned} g(\mathbf{x}|0)_{n+k+1} &= n+k+1, \\ g(\mathbf{x}|0)_i &= \varphi_i \text{ otherwise;} \\ g(\mathbf{x}|1)_r &= n+k+1, \\ g(\mathbf{x}|1)_{n+k+1} &= n+k-4, \\ g(\mathbf{x}|1)_i &= \varphi_i \text{ otherwise;} \end{aligned}$$

if n is odd or $x_n < 2$, then

$$\begin{aligned} g(\mathbf{x}|2)_{n+k} &= n+k+1, \\ g(\mathbf{x}|2)_{n+k+1} &= \varphi_{n+k}, \\ g(\mathbf{x}|2)_i &= \varphi_i \text{ otherwise;} \end{aligned}$$

if n is even and $x_n = 2$, then

$$\begin{aligned} g(\mathbf{x}|2)_{n+k-1} &= n+k+1, \\ g(\mathbf{x}|2)_{n+k+1} &= \varphi_{n+k-1}, \\ g(\mathbf{x}|2)_i &= \varphi_i \text{ otherwise.} \end{aligned}$$

We note that $g(\mathbf{x}|a)_i \neq f(\mathbf{x})_i$ for at most one value of $i \leq n+k$.

For $f \in \mathcal{F}_{m,k}$, we define a sequence of functions $f \in \mathcal{F}_{n,k}$, for all $n \geq m$, recursively by

$$f_m = f \text{ and } f_{n+1} = H(f_n) \text{ for } n \geq m.$$

Theorem 3 *If $f_m \in \mathcal{P}_{m,k}$ where $k \geq 0$, m is odd, and*

$$f_m(\mathbf{x})_{m+k} \notin \{m+k-4, m+k-3\} \text{ for all } \mathbf{x} \in Z_3^m,$$

then $f_n \in \mathcal{P}_{n,k}$ for all $n \geq m$.

Theorem 4 *If $f_m \in \mathcal{I}_{m,k}$, where $k > 0$ and m is odd, and*

$$f_m(\mathbf{x})_{m+k} \notin \{m+k-4, m+k-3\} \text{ for all } \mathbf{x} \in Z_3^m,$$

then $f_n \in \mathcal{I}_{n,k}$ for all $n \geq m$.

Proof: We prove Theorem 4; the proof of Theorem 3 is similar (and a little simpler). The proof is by induction. First we prove that $g = f_{m+1} \in \mathcal{I}_{m+1,k}$. Let $\mathbf{x}, \mathbf{y} \in Z_3^m$ and

$$f(\mathbf{x}) = (\varphi_1, \varphi_2, \dots, \varphi_{m+k}), \quad \varphi_r = m+k-4,$$

$$f(\mathbf{y}) = (\gamma_1, \gamma_2, \dots, \gamma_{m+k}), \quad \gamma_s = m+k-4.$$

We want to show that

$$d_H(g(\mathbf{x}|a), g(\mathbf{y}|b)) > d_H((\mathbf{x}|a), (\mathbf{y}|b))$$

if $(\mathbf{x}|a) \neq (\mathbf{y}|b)$.

First, consider $\mathbf{x} = \mathbf{y}$ and $a \neq b$. Since $\varphi_{m+k} \neq m+k-4$, it follows immediately from the definition of g that

$$d_H(g(\mathbf{x}|a), g(\mathbf{x}|b)) \geq 2 > 1 = d_H((\mathbf{x}|a), (\mathbf{x}|b)).$$

For $\mathbf{x} \neq \mathbf{y}$, we want to show that

$$d_H(g(\mathbf{x}|a), g(\mathbf{y}|b)) - d_H(f(\mathbf{x}), f(\mathbf{y})) \geq d_H(a, b) \quad (2)$$

for all $a, b \in Z_3$ since this implies

$$\begin{aligned} d_H(g(\mathbf{x}|a), g(\mathbf{y}|b)) &\geq d_H(f(\mathbf{x}), f(\mathbf{y})) + d_H(a, b) \\ &> d_H(\mathbf{x}, \mathbf{y}) + d_H(a, b) \\ &= d_H((\mathbf{x}|a), (\mathbf{y}|b)). \end{aligned}$$

The condition (2) is equivalent to the following.

$$\sum_{i=1}^{m+k+1} (\Delta_{g,i} - \Delta_{f,i}) \geq d_H(a, b), \quad (3)$$

where

$$\Delta_{g,i} = d_H(g(\mathbf{x}|a)_i, g(\mathbf{y}|b)_i)$$

and

$$\Delta_{f,i} = d_H(f(\mathbf{x})_i, f(\mathbf{y})_i),$$

and where, for technical reasons, we define

$$\Delta_{f,n+k+1} = 0.$$

The point is at most three of the terms $\Delta_{g,i} - \Delta_{f,i}$ are non-zero. We look at one combination of a and b in detail as an illustration, namely $a = 1$ and $b = 2$. Then $g(\mathbf{x}|a)_i = f(\mathbf{x})_i$ and $g(\mathbf{y}|b)_i = f(\mathbf{y})_i$ and so $\Delta_{g,i} = \Delta_{f,i}$ for all $i \leq m + k + 1$, *except* in the following three cases

| i | $f(\mathbf{x})_i$ | $f(\mathbf{y})_i$ | $g(\mathbf{x} a)_i$ | $g(\mathbf{y} b)_i$ |
|-------------|-------------------|-------------------|---------------------|---------------------|
| r | $m + k - 4$ | γ_r | $m + k + 1$ | γ_r |
| $m + k$ | φ_{m+k} | γ_{m+k} | φ_{m+k} | $m + k + 1$ |
| $m + k + 1$ | $-$ | $-$ | $m + k - 4$ | γ_{m+k} |

| i | $\Delta_{f,i}$ | $\Delta_{g,i}$ | $\Delta_{g,i} - \Delta_{f,i}$ |
|-------------|----------------|----------------|-------------------------------|
| r | 0 or 1 | 1 | 0 or 1 |
| $m + k$ | 0 or 1 | 1 | 0 or 1 |
| $m + k + 1$ | 0 | 1 | 1 |

Note that we have used the fact that $\gamma_{m+k} \neq m + k - 4$. We see that $\sum(\Delta_{g,i} - \Delta_{f,i}) \geq 1 = d_H(a, b)$.

The other combinations of a and b are similar. This proves that $f_{m+1} = g \in \mathcal{I}_{m+1,k}$.

Now, let $h = H(g) = f_{m+2}$. A similar analysis will show that $h \in \mathcal{I}_{m+2,k}$. We first give a table of the last three symbols in $h(\mathbf{x}|a_1a_2)$ as these three symbols are the most important in the proof. Let $\varphi_s = m + k - 3$. By assumption, $s < m + k$.

| a_1a_2 | $h(\mathbf{x} a_1a_2)_{m+k}$ | $h(\mathbf{x} a_1a_2)_{m+k+1}$ | $h(\mathbf{x} a_1a_2)_{m+k+2}$ |
|----------|------------------------------|--------------------------------|--------------------------------|
| 00 | φ_{m+k} | $m + k + 1$ | $m + k + 2$ |
| 10 | φ_{m+k} | $m + k - 4$ | $m + k + 2$ |
| 20 | $m + k + 1$ | φ_{m+k} | $m + k + 2$ |
| 01 | φ_{m+k} | $m + k + 1$ | $m + k - 3$ |
| 11 | φ_{m+k} | $m + k - 4$ | $m + k - 3$ |
| 21 | $m + k + 1$ | φ_{m+k} | $m + k - 3$ |
| 02 | φ_{m+k} | $m + k + 2$ | $m + k + 1$ |
| 12 | φ_{m+k} | $m + k + 2$ | $m + k - 4$ |
| 22 | $m + k + 2$ | φ_{m+k} | $m + k + 1$ |

In addition,

$$h(\mathbf{x}|1a_2)_r = m + k + 1 \text{ and } h(\mathbf{x}|a_11)_s = m + k + 2.$$

Note that we have used the fact that $\varphi_{m+k} \neq m + k - 3$ here, since if we had $\varphi_{m+k} = m + k - 3$, then we would for example have had $h(\mathbf{x}|01)_{m+k} = m + k + 2$. From the table we first see that

$$d_H(h(\mathbf{x}|a_1a_2), h(\mathbf{x}|b_1b_2)) > d_H(a_1a_2, b_1b_2)$$

if $a_1a_2 \neq b_1b_2$. For example $h(\mathbf{x}|10)$ and $h(\mathbf{x}|21)$ differ in positions $r, s, m+k, m+k+1$ and $m+k+2$. As another example, $h(\mathbf{x}|02)$ and $h(\mathbf{x}|22)$ differ in positions $m+k$ and $m+k+1$.

Next, consider $d_H(h(\mathbf{x}|a_1a_2), h(\mathbf{y}|b_1b_2))$ for $\mathbf{x} \neq \mathbf{y}$. We see that

$$d_H(h(\mathbf{x}|a_1a_2)_i, h(\mathbf{y}|b_1b_2)_i) \geq d_H(f(\mathbf{x})_i, f(\mathbf{y})_i)$$

for $i < m+k$: from the table above, we can see that

$$\begin{aligned} & d_H(h(\mathbf{x}|a_1a_2)_{m+k}h(\mathbf{x}|a_1a_2)_{m+k+1}h(\mathbf{x}|a_1a_2)_{m+k+2}, \\ & \quad h(\mathbf{y}|b_1b_2)_{m+k}h(\mathbf{y}|b_1b_2)_{m+k+1}h(\mathbf{y}|b_1b_2)_{m+k+2}) \\ & \geq d_H(\varphi_{m+k}, \gamma_{m+k}) + d_H(a_1a_2, b_1b_2). \end{aligned}$$

As an example, let $a_1a_2 = 10$ and $b_1b_2 = 02$. Then

$$\begin{aligned} & h(\mathbf{x}|10)_{m+k}, h(\mathbf{x}|10)_{m+k+1}, h(\mathbf{x}|10)_{m+k+2} \\ & = \varphi_{m+k}, m+k-4, m+k+2 \end{aligned}$$

and

$$\begin{aligned} & h(\mathbf{y}|02)_{m+k}, h(\mathbf{y}|02)_{m+k+1}, h(\mathbf{y}|02)_{m+k+2} \\ & = \gamma_{m+k}, m+k+2, m+k+1. \end{aligned}$$

The distance between the two is 2 (if $\varphi_{m+k} = \gamma_{m+k}$) or 3 (otherwise). The other combinations of a_1a_2 and b_1b_2 are similar. From this we can conclude that $h \in \mathcal{I}_{m+2,k}$ in a similar way we showed that $g \in \mathcal{I}_{m+1,k}$ above.

Further, we note that

$$h(\mathbf{x}|a_1a_2)_{m+k+2} \notin \{(m+2) + k - 4, (m+2) + k - 3\}.$$

Therefore, we can repeat the argument and, by induction, obtain $f_n \in \mathcal{I}_{n,k}$ for all $n \geq m$.

A function F is given by an explicit listing in the appendix. It belongs to $\mathcal{I}_{3,2}$ and satisfy $F(\mathbf{x})_5 \notin \{1, 2\}$. This, combined with Theorem 4, proves Theorem 1 a).

4 Proof of Theorem 1, second part

To prove Theorem 1 b), using Theorem 3, we need some $f \in \mathcal{P}_{9,1}$ such that

$$f(\mathbf{x})_{10} \notin \{6, 7\} \text{ for all } \mathbf{x} \in Z_3^9. \quad (4)$$

An extensive computer search has been unsuccessful in coming up with such a mapping. However, an indirect approach has been successful. The approach is to construct f from two simpler mappings found by computer search.

For a vector $\rho = (\rho_1, \rho_2, \dots, \rho_n)$ and a set $X \subset \{1, 2, \dots, n\}$, let $\rho_{\setminus X}$ denote the vector obtained from ρ by removing the elements with subscript in X . For example,

$$(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6)_{\setminus \{1,5\}} = (\rho_2, \rho_3, \rho_4, \rho_6).$$

By computer search we have found mappings $G \in \mathcal{F}_{5,2}$ and $H \in \mathcal{F}_{4,2}$ that satisfy the following conditions

- a) for every $\mathbf{x} \in Z_3^5$, $6 \in \{G(\mathbf{x})_1, G(\mathbf{x})_2, G(\mathbf{x})_3\}$,
- b) for every $\mathbf{x} \in Z_3^5$, $7 \in \{G(\mathbf{x})_4, G(\mathbf{x})_5, G(\mathbf{x})_6\}$,
- c) for every distinct $\mathbf{x}, \mathbf{y} \in Z_3^5$,
 $d_H(G(\mathbf{x})_{\setminus \{7\}}, G(\mathbf{y})_{\setminus \{7\}}) \geq d_H(\mathbf{x}, \mathbf{y})$,
- d) for every $\mathbf{u} \in Z_3^4$, $1 \in \{H(\mathbf{u})_1, H(\mathbf{u})_2, H(\mathbf{u})_3\}$,
- e) for every distinct $\mathbf{u}, \mathbf{v} \in Z_3^4$,
 $d_H(H(\mathbf{u})_{\setminus \{5,6\}}, H(\mathbf{v})_{\setminus \{5,6\}}) \geq d_H(\mathbf{u}, \mathbf{v})$.

The mappings G and H are listed explicitly in the appendix. We will now show how these mappings can be combined to produce a mapping $f \in \mathcal{P}_{9,1}$ satisfying (4).

Let $\mathbf{x} \in Z_3^9$. Then $\mathbf{x} = (\mathbf{x}_L, \mathbf{x}_R)$, where $\mathbf{x}_L \in Z_3^5$ and $\mathbf{x}_R \in Z_3^4$. Let

$$\begin{aligned} (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6, \varphi_7) &= G(\mathbf{x}_L), \\ (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6) &= H(\mathbf{x}_R) + (4, 4, 4, 4, 4, 4). \end{aligned}$$

We note that Condition d) implies that $\gamma_5 \geq 6$ and $\gamma_6 \geq 6$. Similarly, Conditions a) and b) imply that $\varphi_7 \leq 5$.

Define $\rho = (\rho_1, \rho_2, \dots, \rho_{10})$ as follows.

$$\begin{aligned} \rho_i &= \gamma_5 & \text{if } 1 \leq i \leq 3 & \quad \text{and} \quad \varphi_i = 6, \\ \rho_i &= \gamma_6 & \text{if } 4 \leq i \leq 6 & \quad \text{and} \quad \varphi_i = 7, \\ \rho_i &= \varphi_i & \text{if } 1 \leq i \leq 6 & \quad \text{and} \quad \varphi_i \leq 5, \\ \rho_i &= \varphi_7 & \text{if } 7 \leq i \leq 9 & \quad \text{and} \quad \gamma_{i-6} = 5, \\ \rho_i &= \gamma_{i-6} & \text{if } 7 \leq i \leq 10 & \quad \text{and} \quad \gamma_{i-6} \geq 6. \end{aligned}$$

In ρ , swap 1 and 6 and also swap 2 and 7, and let the resulting array be denoted by π . More formally,

$$\begin{aligned}\pi_i &= 1 \text{ if } \rho_i = 6, \\ \pi_i &= 2 \text{ if } \rho_i = 7, \\ \pi_i &= 6 \text{ if } \rho_i = 1, \\ \pi_i &= 7 \text{ if } \rho_i = 2, \\ \pi_i &= \rho_i \text{ otherwise.}\end{aligned}$$

Then define

$$f(\mathbf{x}) = \pi.$$

We will show that f has the stated properties. We first show that $\pi \in S_{10}$. We have $\varphi \in S_7$ and γ is a permutation of $(5, 6, 7, 8, 9, 10)$. In particular, 5, 6, and 7 appear both in φ and γ . The effect of the first line in the definition of ρ is to move another elements (γ_5) into the position where φ has a 6. Similarly, the second line overwrites the 7 in ρ , and the fourth line overwrites the 5 in γ . The definition of ρ is then the concatenation of the six first (overwritten) elements of φ and the five first (overwritten) elements of γ . Therefore, ρ contains no duplicate elements, that is, $\rho \in S_{10}$.

The element 1 in ρ must be either in one of the first six positions, coming from φ , or in one of the positions 7 – 9 (if $\varphi_7 = 1$). Similarly, the element 2 must be in one of the first nine positions of ρ . Therefore, both 6 and 7 must be among the first nine elements of π , that is $\pi_{10} \notin \{6, 7\}$.

Finally, we must show that f is distance preserving. Let $\mathbf{x} \neq \mathbf{x}'$, and let the arrays corresponding to \mathbf{x}' be denoted by φ' , γ' , ρ' and π' . By assumption,

$$\begin{aligned}d_H(\mathbf{x}, \mathbf{x}') &= d_H(\mathbf{x}_L, \mathbf{x}'_L) + d_H(\mathbf{x}_R, \mathbf{x}'_R) \\ &\leq d_H(\varphi_{\setminus\{7\}}, \varphi'_{\setminus\{7\}}) + d_H(\gamma_{\setminus\{5,6\}}, \gamma'_{\setminus\{5,6\}}).\end{aligned}\tag{5}$$

For $1 \leq i \leq 6$ we have

$$d_H(\varphi_i, \varphi'_i) \leq d_H(\rho_i, \rho'_i).\tag{6}$$

If $\varphi_i = \varphi'_i$ this is obvious. Otherwise, we may assume without loss of generality that $\varphi'_i < \varphi_i$ and we must show that $\rho_i \neq \rho'_i$. If $\varphi_i \leq 5$, then

$$\rho'_i = \varphi'_i < \varphi_i = \rho_i.$$

If $\varphi_i = 6$, then

$$\rho'_i = \varphi'_i \leq 5 \text{ and } \rho_i = \gamma_5 \geq 6.$$

If $\varphi_i = 7$, then $4 \leq i \leq 6$ and so $\varphi'_i \neq 6$. Hence

$$\rho'_i = \varphi'_i \leq 5 \text{ and } \rho_i = \gamma_6 \geq 6.$$

This completes that proof of (6). A similar arguments show that for $7 \leq i \leq 10$ we have

$$d_H(\gamma_{i-6}, \gamma'_{i-6}) \leq d_H(\rho_i, \rho'_i), \quad (7)$$

and that for $1 \leq i \leq 10$ we have

$$d_H(\rho_i, \rho'_i) \leq d_H(\pi_i, \pi'_i). \quad (8)$$

Combining (5)–(8), we get

$$\begin{aligned} d_H(\mathbf{x}, \mathbf{x}') &\leq d_H(\varphi_{\setminus\{7\}}, \varphi'_{\setminus\{7\}}) + d_H(\gamma_{\setminus\{5,6\}}, \gamma'_{\setminus\{5,6\}}) \\ &\leq d_H(\rho, \rho') \leq d_H(\pi, \pi'). \end{aligned}$$

Hence, f is distance preserving.

5 Proof of Theorem 1, last part

The construction of a mapping $f \in \mathcal{P}_{13,0}$ which proves Theorem 1 c) is similar to the construction in the previous section. However, the construction is more involved and contains several steps. We will describe the constructions and properties of the intermediate mappings. The details of proofs are similar to the proof in the previous section and we omit these details.

We start with three mappings $R, S \in \mathcal{F}_{3,2}$ and $T \in \mathcal{F}_{4,2}$. These were found by computer search and are listed explicitly in the appendix. They have the following properties:

- for every $\mathbf{x} \in Z_3^3$, $1 \in \{R(\mathbf{x})_1, R(\mathbf{x})_2, R(\mathbf{x})_3\}$,
- for every $\mathbf{x} \in Z_3^3$, $R(\mathbf{x})_5 \neq 5$,
- for every distinct $\mathbf{x}, \mathbf{y} \in Z_3^3$,
 $d_H(R(\mathbf{x})_{\setminus\{4,5\}}, R(\mathbf{y})_{\setminus\{4,5\}}) \geq d_H(\mathbf{x}, \mathbf{y})$,
- for every $\mathbf{x} \in Z_3^3$, $2 \in \{S(\mathbf{x})_1, S(\mathbf{x})_2, S(\mathbf{x})_3\}$,
- for every $\mathbf{x} \in Z_3^3$, $S(\mathbf{x})_5 \neq 1$,
- for every distinct $\mathbf{x}, \mathbf{y} \in Z_3^3$,
 $d_H(S(\mathbf{x})_{\setminus\{4,5\}}, S(\mathbf{y})_{\setminus\{4,5\}}) \geq d_H(\mathbf{x}, \mathbf{y})$,

- for every $\mathbf{x} \in Z_3^4$, $2 \in \{T(\mathbf{x})_1, T(\mathbf{x})_2, T(\mathbf{x})_3\}$,
- for every $\mathbf{x} \in Z_3^4$, $T(\mathbf{x})_6 \neq 1$,
- for every distinct $\mathbf{x}, \mathbf{y} \in Z_3^4$,
 $d_H(T(\mathbf{x})_{\setminus\{5,6\}}, T(\mathbf{y})_{\setminus\{5,6\}}) \geq d_H(\mathbf{x}, \mathbf{y})$.

These mappings are used as building blocks similarly to what was done in the previous section.

Construction of $U \in \mathcal{F}_{6,2}$

Let $\mathbf{x} \in Z_3^6$ and let

$$\begin{aligned} (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5) &= R(x_1, x_2, x_3), \\ (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) &= S(x_4, x_5, x_6) + (3, 3, 3, 3, 3). \end{aligned}$$

Define $\rho = (\rho_1, \rho_2, \dots, \rho_8)$ as follows.

$$\begin{aligned} \rho_i &= \gamma_5 & \text{if } 1 \leq i \leq 4 & \text{ and } \varphi_i = 5, \\ \rho_i &= \varphi_i & \text{if } 1 \leq i \leq 4 & \text{ and } \varphi_i \neq 5, \\ \rho_i &= \varphi_5 & \text{if } 5 \leq i \leq 8 & \text{ and } \gamma_{i-4} = 4, \\ \rho_i &= \gamma_{i-4} & \text{if } 5 \leq i \leq 8 & \text{ and } \gamma_{i-4} \neq 4. \end{aligned}$$

In ρ , swap 1 and 7 and also swap 5 and 8, and let the resulting array be $U(\mathbf{x})$. It has the following properties:

- for every $\mathbf{x} \in Z_3^6$, $7 \in \{U(\mathbf{x})_1, U(\mathbf{x})_2, U(\mathbf{x})_3\}$,
- for every $\mathbf{x} \in Z_3^6$, $8 \in \{U(\mathbf{x})_5, U(\mathbf{x})_6, U(\mathbf{x})_7\}$,
- for every distinct $\mathbf{x}, \mathbf{y} \in Z_3^6$,
 $d_H(U(\mathbf{x})_{\setminus\{4,8\}}, U(\mathbf{y})_{\setminus\{4,8\}}) \geq d_H(\mathbf{x}, \mathbf{y})$.

Construction of $V \in \mathcal{F}_{7,2}$

Let $\mathbf{x} \in Z_3^7$ and let

$$\begin{aligned} (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5) &= R(x_1, x_2, x_3), \\ (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6) &= T(x_4, x_5, x_6, x_7) + (3, 3, \dots, 3). \end{aligned}$$

Define $\rho = (\rho_1, \rho_2, \dots, \rho_8, \rho_9)$ as follows.

$$\begin{aligned} \rho_i &= \gamma_6 & \text{if } 1 \leq i \leq 4 & \text{ and } \varphi_i = 5, \\ \rho_i &= \varphi_i & \text{if } 1 \leq i \leq 4 & \text{ and } \varphi_i \neq 5, \\ \rho_i &= \varphi_5 & \text{if } 5 \leq i \leq 9 & \text{ and } \gamma_{i-4} = 4, \\ \rho_i &= \gamma_{i-4} & \text{if } 5 \leq i \leq 9 & \text{ and } \gamma_{i-4} \neq 4. \end{aligned}$$

In ρ , swap 2 and 5, and let the resulting array be $V(\mathbf{x})$. It has the following properties:

- for every $\mathbf{x} \in Z_3^7$, $1 \in \{V(\mathbf{x})_1, V(\mathbf{x})_2, V(\mathbf{x})_3\}$,
- for every $\mathbf{x} \in Z_3^7$, $2 \in \{V(\mathbf{x})_5, V(\mathbf{x})_6, V(\mathbf{x})_7\}$,
- for every distinct $\mathbf{x}, \mathbf{y} \in Z_3^7$,
 $d_H(V(\mathbf{x})_{\setminus\{4,9\}}, V(\mathbf{y})_{\setminus\{4,9\}}) \geq d_H(\mathbf{x}, \mathbf{y})$.

Construction of $f \in \mathcal{P}_{13,0}$

Let $\mathbf{x} \in Z_3^{13}$ and let

$$\begin{aligned} (\varphi_1, \varphi_2, \dots, \varphi_8) &= U(x_1, x_2, \dots, x_6), \\ (\gamma_1, \gamma_2, \dots, \gamma_9) &= V(x_7, x_8, \dots, x_{13}) + (4, 4, \dots, 4). \end{aligned}$$

Define $\rho = (\rho_1, \rho_2, \dots, \rho_{13})$ as follows.

$$\begin{aligned} \rho_i &= \gamma_4 & \text{if } 1 \leq i \leq 3 & \text{ and } \varphi_i = 7, \\ \rho_i &= \varphi_i & \text{if } 1 \leq i \leq 3 & \text{ and } \varphi_i \neq 7, \\ \rho_i &= \gamma_9 & \text{if } 4 \leq i \leq 6 & \text{ and } \varphi_{i+1} = 8, \\ \rho_i &= \varphi_{i+1} & \text{if } 4 \leq i \leq 6 & \text{ and } \varphi_{i+1} \neq 8, \\ \rho_i &= \varphi_4 & \text{if } 7 \leq i \leq 9 & \text{ and } \gamma_{i-6} = 5, \\ \rho_i &= \gamma_{i-6} & \text{if } 7 \leq i \leq 9 & \text{ and } \gamma_{i-6} \neq 5, \\ \rho_i &= \varphi_8 & \text{if } 10 \leq i \leq 13 & \text{ and } \gamma_{i-5} = 6, \\ \rho_i &= \gamma_{i-5} & \text{if } 10 \leq i \leq 13 & \text{ and } \gamma_{i-5} \neq 6. \end{aligned}$$

In ρ , swap 1 and 9 and also swap 2 and 10, and let the resulting array be $f(\mathbf{x})$. Then

$$f \in \mathcal{P}_{13,0} \text{ and } f(\mathbf{x})_{13} \notin \{9, 10\}.$$

References

- [1] A.E. Brouwer, Heikki O. Hämäläinen, Patric R.J. Östergård, N.J.A. Sloane, “Bounds on mixed binary/ternary codes”, *IEEE Trans. on Inform. Theory*, vol. 44, no. 1, pp. 140–161, Jan. 1998.
- [2] J.-C. Chang, “Distance-increasing mappings from binary vectors to permutations”, *IEEE Trans. on Inform. Theory*, vol. 51, no. 1, pp. 359–363, Jan. 2005.
- [3] J.-C. Chang, “New algorithms of distance-increasing mappings from binary vectors to permutations by swaps”, *Designs, Codes and Cryptography*, vol. 39, pp. 335–345, Jan. 2006.
- [4] J.-C. Chang, “Distance-increasing mappings from binary vectors to permutations that increase Hamming distances by at least two”, *IEEE Trans. on Inform. Theory*, vol. 52, no. 4, pp. 1683–1689, April 2006.
- [5] J.-C. Chang, R.-J. Chen, T. Kløve, and S.-C. Tsai, “Distance-preserving mappings from binary vectors to permutations”, *IEEE Trans. on Inform. Theory*, vol. 49, pp. 1054–1059, Apr. 2003.
- [6] C. J. Colbourn, T. Kløve, and A. C. H. Ling, “Permutation arrays for powerline communication and mutually orthogonal latin squares”, *IEEE Trans. on Inform. Theory*, vol. 50, no. 6, pp. 1289–1291, June 2004.
- [7] C. Ding, F.-W. Fu, T. Kløve, and V. K. Wei, “Constructions of permutation arrays”, *IEEE Trans. on Inform. Theory*, vol. 48, pp. 977–980, Apr. 2002.
- [8] H. C. Ferreira and A. J. H. Vinck, “Inference cancellation with permutation trellis arrays”, *Proc. IEEE Vehicular Technology Conf.*, pp. 2401–2407, 2000.
- [9] H. C. Ferreira, A. J. H. Vinck, T. G. Swart, and A. L. Nel, “Permutation trellis codes”, *Proc. IEEE Trans. on Communications*, vol. 53, no. 11, pp. 1782–1789, Nov. 2005.
- [10] H. C. Ferreira, D Wright, and A. L. Nel, “Hamming distance preserving mappings and trellis codes with constrained binary symbols”, *IEEE Trans. on Inform. Theory*, vol. 35, no. 5, pp. 1098–1103, Sept. 1989.

- [11] F.-W. Fu and T. Kløve, “Two constructions of permutation arrays”, *IEEE Trans. on Inform. Theory*, vol. 50, pp. 881–883, May. 2004.
- [12] Y.-Y. Huang, S.-C. Tsai, H.-L. Wu, “On the construction of permutation arrays via mappings from binary vectors to permutations”, *Designs, Codes and Cryptography*, vol. 40, pp. 139–155, 2006.
- [13] T. Kløve, “Classification of permutation codes of length 6 and minimum distance 5”, *Proc. Int. Symp. Information Theory and Its Applications*, 2000, pp. 465–468.
- [14] K. Lee, “New distance-preserving maps of odd length”, *IEEE Trans. on Inform. Theory*, vol. 50, no. 10, pp. 2539–2543, Oct. 2004.
- [15] K. Lee, “Cyclic constructions of distance-preserving maps”, *IEEE Trans. on Inform. Theory*, vol. 51, no. 12, pp. 4292–4396, Dec. 2005.
- [16] K. Lee, “Distance-increasing maps of all length by simple mapping algorithms”, *arXiv:cs.IT/0509073*, 23 Sept. 2005.
- [17] J.-S. Lin, J.-C. Chang, and R.-J. Chen, “New simple constructions of distance-increasing mappings from binary vectors to permutations”, *Information Processing Letters*, vol. 100, no. 2, pp. 83–89, Oct. 2006.
- [18] V. S. Pless and W. C. Huffman, Eds., *Handbook of Coding Theory*. Amsterdam, The Netherlands: Elsevier, 1998.
- [19] T. G Swart, I. de Beer, H. C. Ferreira, “On the distance optimality of permutation mappings”, *Proc. IEEE Int. Symp. Information Theory*, Adelaide, Australia, September 2005, pp. 1068–1072.
- [20] T. G Swart and H. C. Ferreira, “A multilevel construction for mappings from binary sequences to permutation sequences”, *Proc. IEEE Int. Symp. Information Theory*, Seattle, USA, July 2006, pp. 1895–1899.
- [21] A. J. H. Vinck, “Coding and modulation for powerline communications”, *A.E.Ü. Int. J. Electron. Commun.*, vol. 54, no. 1, pp. 45–49, Oct. 2000.
- [22] A. J. H. Vinck and J. Häring, “Coding and modulation for power-line communications”, *Proc. Int. Symp. Power Line Communication*, Limerick, Ireland, Apr. 57, 2000.

- [23] A. J. H. Vinck, J. Häring, and T. Wadayama, “Coded M-FSK for power-line communications”, *Proc. IEEE Int. Symp. Information Theory*, Sorrento, Italy, June 2000, p. 137.
- [24] T. Wadayama and A. J. H. Vinck, “A multilevel construction of permutation codes”, *IEICE Trans. Fundamentals Electron., Commun. Comp. Sci.*, vol. 84, pp. 2518–2522, 2001.

Appendix

Listing of the elements $\mathbf{x} \in Z_3^3$ and the corresponding values of $F(\mathbf{x}) \in S_5$.

(0,0,0)(1,2,3,4,5), (0,0,1)(1,2,5,4,3), (0,0,2)(1,2,3,5,4),
 (0,1,0)(4,2,3,1,5), (0,1,1)(4,2,5,1,3), (0,1,2)(5,2,3,1,4),
 (0,2,0)(1,4,3,2,5), (0,2,1)(1,4,5,2,3), (0,2,2)(1,5,3,2,4),
 (1,0,0)(2,3,1,4,5), (1,0,1)(2,5,1,4,3), (1,0,2)(2,3,1,5,4),
 (1,1,0)(2,3,4,1,5), (1,1,1)(2,5,4,1,3), (1,1,2)(2,3,5,1,4),
 (1,2,0)(4,3,1,2,5), (1,2,1)(4,5,1,2,3), (1,2,2)(5,3,1,2,4),
 (2,0,0)(3,1,2,4,5), (2,0,1)(5,1,2,4,3), (2,0,2)(3,1,2,5,4),
 (2,1,0)(3,4,2,1,5), (2,1,1)(5,4,2,1,3), (2,1,2)(3,5,2,1,4),
 (2,2,0)(3,1,4,2,5), (2,2,1)(5,1,4,2,3), (2,2,2)(3,1,5,2,4)

Listing of the elements $\mathbf{x} \in Z_3^5$ and the corresponding values of $G(\mathbf{x}) \in S_7$.

(0,0,0,0,0)(6,1,2,7,3,4,5), (0,0,0,0,1)(6,3,2,7,1,5,4),
 (0,0,0,0,2)(6,3,2,7,4,5,1), (0,0,0,1,0)(6,2,1,7,5,3,4),
 (0,0,0,1,1)(6,1,2,7,5,3,4), (0,0,0,1,2)(6,3,2,7,5,4,1),
 (0,0,0,2,0)(6,1,2,7,3,5,4), (0,0,0,2,1)(6,3,1,7,2,5,4),
 (0,0,0,2,2)(6,3,1,7,4,5,2), (0,0,1,0,0)(6,2,5,7,1,4,3),
 (0,0,1,0,1)(6,2,5,7,3,4,1), (0,0,1,0,2)(6,3,5,7,4,1,2),
 (0,0,1,1,0)(6,2,5,7,1,3,4), (0,0,1,1,1)(6,5,1,7,2,3,4),
 (0,0,1,1,2)(6,2,5,7,4,3,1), (0,0,1,2,0)(6,4,5,7,1,3,2),
 (0,0,1,2,1)(6,2,4,7,1,5,3), (0,0,1,2,2)(6,1,5,7,4,2,3),
 (0,0,2,0,0)(6,4,2,7,3,1,5), (0,0,2,0,1)(6,3,4,7,2,1,5),
 (0,0,2,0,2)(6,3,2,7,4,1,5), (0,0,2,1,0)(6,4,1,7,5,3,2),
 (0,0,2,1,1)(6,5,4,7,2,3,1), (0,0,2,1,2)(6,5,2,7,4,3,1),
 (0,0,2,2,0)(6,4,1,7,5,2,3), (0,0,2,2,1)(6,5,4,7,3,2,1),
 (0,0,2,2,2)(6,5,3,7,4,2,1), (0,1,0,0,0)(6,1,3,2,7,5,4),
 (0,1,0,0,1)(6,3,2,4,7,5,1), (0,1,0,0,2)(6,3,2,5,7,4,1),
 (0,1,0,1,0)(6,4,2,5,7,3,1), (0,1,0,1,1)(6,2,1,5,7,3,4),

$(0,1,0,1,2)(6,5,2,1,7,4,3)$, $(0,1,0,2,0)(6,2,1,3,7,5,4)$,
 $(0,1,0,2,1)(6,3,1,4,7,5,2)$, $(0,1,0,2,2)(6,5,2,3,7,4,1)$,
 $(0,1,1,0,0)(6,3,5,2,7,1,4)$, $(0,1,1,0,1)(6,2,3,5,7,4,1)$,
 $(0,1,1,0,2)(6,3,5,2,7,4,1)$, $(0,1,1,1,0)(6,2,5,4,7,3,1)$,
 $(0,1,1,1,1)(6,2,5,1,7,3,4)$, $(0,1,1,1,2)(6,3,5,1,7,4,2)$,
 $(0,1,1,2,0)(6,2,5,3,7,1,4)$, $(0,1,1,2,1)(6,5,1,3,7,2,4)$,
 $(0,1,1,2,2)(6,4,5,3,7,2,1)$, $(0,1,2,0,0)(6,5,4,2,7,1,3)$,
 $(0,1,2,0,1)(6,4,3,2,7,1,5)$, $(0,1,2,0,2)(6,5,3,2,7,4,1)$,
 $(0,1,2,1,0)(6,4,2,1,7,3,5)$, $(0,1,2,1,1)(6,5,4,1,7,3,2)$,
 $(0,1,2,1,2)(6,3,4,5,7,2,1)$, $(0,1,2,2,0)(6,5,4,3,7,1,2)$,
 $(0,1,2,2,1)(6,5,4,3,7,2,1)$, $(0,1,2,2,2)(6,4,3,1,7,2,5)$,
 $(0,2,0,0,0)(6,4,1,5,3,7,2)$, $(0,2,0,0,1)(6,3,2,4,1,7,5)$,
 $(0,2,0,0,2)(6,3,2,5,4,7,1)$, $(0,2,0,1,0)(6,1,4,2,5,7,3)$,
 $(0,2,0,1,1)(6,2,4,1,5,7,3)$, $(0,2,0,1,2)(6,3,2,1,5,7,4)$,
 $(0,2,0,2,0)(6,4,2,3,5,7,1)$, $(0,2,0,2,1)(6,1,2,3,5,7,4)$,
 $(0,2,0,2,2)(6,3,1,5,4,7,2)$, $(0,2,1,0,0)(6,3,5,2,1,7,4)$,
 $(0,2,1,0,1)(6,5,1,4,2,7,3)$, $(0,2,1,0,2)(6,3,5,2,4,7,1)$,
 $(0,2,1,1,0)(6,1,5,4,2,7,3)$, $(0,2,1,1,1)(6,2,5,1,3,7,4)$,
 $(0,2,1,1,2)(6,4,5,1,2,7,3)$, $(0,2,1,2,0)(6,2,5,3,1,7,4)$,
 $(0,2,1,2,1)(6,5,1,3,2,7,4)$, $(0,2,1,2,2)(6,3,5,1,4,7,2)$,
 $(0,2,2,0,0)(6,5,3,4,1,7,2)$, $(0,2,2,0,1)(6,5,1,4,3,7,2)$,
 $(0,2,2,0,2)(6,5,3,2,4,7,1)$, $(0,2,2,1,0)(6,4,2,1,3,7,5)$,
 $(0,2,2,1,1)(6,5,4,1,3,7,2)$, $(0,2,2,1,2)(6,5,3,1,4,7,2)$,
 $(0,2,2,2,0)(6,5,4,3,1,7,2)$, $(0,2,2,2,1)(6,5,4,3,2,7,1)$,
 $(0,2,2,2,2)(6,5,2,3,4,7,1)$, $(1,0,0,0,0)(2,6,1,7,3,5,4)$,
 $(1,0,0,0,1)(1,6,3,7,2,5,4)$, $(1,0,0,0,2)(3,6,2,7,1,4,5)$,
 $(1,0,0,1,0)(4,6,1,7,5,3,2)$, $(1,0,0,1,1)(4,6,2,7,5,3,1)$,
 $(1,0,0,1,2)(1,6,2,7,5,3,4)$, $(1,0,0,2,0)(4,6,1,7,5,2,3)$,
 $(1,0,0,2,1)(4,6,1,7,3,2,5)$, $(1,0,0,2,2)(3,6,4,7,1,2,5)$,
 $(1,0,1,0,0)(2,6,5,7,4,1,3)$, $(1,0,1,0,1)(2,6,3,7,1,4,5)$,
 $(1,0,1,0,2)(1,6,3,7,5,4,2)$, $(1,0,1,1,0)(2,6,5,7,1,3,4)$,
 $(1,0,1,1,1)(4,6,5,7,2,3,1)$, $(1,0,1,1,2)(1,6,2,7,4,3,5)$,
 $(1,0,1,2,0)(4,6,5,7,1,2,3)$, $(1,0,1,2,1)(4,6,5,7,3,2,1)$,
 $(1,0,1,2,2)(1,6,5,7,4,3,2)$, $(1,0,2,0,0)(2,6,4,7,3,1,5)$,
 $(1,0,2,0,1)(5,6,3,7,2,1,4)$, $(1,0,2,0,2)(5,6,3,7,4,1,2)$,
 $(1,0,2,1,0)(2,6,4,7,5,1,3)$, $(1,0,2,1,1)(2,6,4,7,5,3,1)$,
 $(1,0,2,1,2)(5,6,3,7,4,2,1)$, $(1,0,2,2,0)(3,6,1,7,5,2,4)$,
 $(1,0,2,2,1)(1,6,5,7,3,2,4)$, $(1,0,2,2,2)(5,6,1,7,4,2,3)$,

$(1,1,0,0,0)(3,6,5,4,7,1,2)$, $(1,1,0,0,1)(3,6,2,5,7,4,1)$,
 $(1,1,0,0,2)(3,6,5,2,7,4,1)$, $(1,1,0,1,0)(4,6,1,2,7,3,5)$,
 $(1,1,0,1,1)(3,6,2,4,7,5,1)$, $(1,1,0,1,2)(4,6,3,1,7,5,2)$,
 $(1,1,0,2,0)(4,6,1,2,7,5,3)$, $(1,1,0,2,1)(4,6,1,3,7,5,2)$,
 $(1,1,0,2,2)(4,6,2,3,7,5,1)$, $(1,1,1,0,0)(4,6,5,2,7,1,3)$,
 $(1,1,1,0,1)(5,6,2,4,7,1,3)$, $(1,1,1,0,2)(4,6,3,5,7,1,2)$,
 $(1,1,1,1,0)(5,6,2,1,7,3,4)$, $(1,1,1,1,1)(4,6,5,1,7,3,2)$,
 $(1,1,1,1,2)(3,6,4,1,7,5,2)$, $(1,1,1,2,0)(4,6,5,3,7,1,2)$,
 $(1,1,1,2,1)(4,6,5,3,7,2,1)$, $(1,1,1,2,2)(5,6,1,3,7,4,2)$,
 $(1,1,2,0,0)(5,6,3,2,7,1,4)$, $(1,1,2,0,1)(5,6,3,4,7,1,2)$,
 $(1,1,2,0,2)(5,6,3,2,7,4,1)$, $(1,1,2,1,0)(5,6,4,1,7,3,2)$,
 $(1,1,2,1,1)(5,6,3,4,7,2,1)$, $(1,1,2,1,2)(5,6,4,1,7,2,3)$,
 $(1,1,2,2,0)(5,6,4,2,7,3,1)$, $(1,1,2,2,1)(5,6,4,3,7,1,2)$,
 $(1,1,2,2,2)(5,6,4,3,7,2,1)$, $(1,2,0,0,0)(3,6,5,4,1,7,2)$,
 $(1,2,0,0,1)(4,6,1,5,3,7,2)$, $(1,2,0,0,2)(3,6,5,2,4,7,1)$,
 $(1,2,0,1,0)(4,6,1,2,5,7,3)$, $(1,2,0,1,1)(4,6,2,1,5,7,3)$,
 $(1,2,0,1,2)(4,6,3,1,5,7,2)$, $(1,2,0,2,0)(4,6,1,3,5,7,2)$,
 $(1,2,0,2,1)(4,6,2,3,5,7,1)$, $(1,2,0,2,2)(5,6,1,3,4,7,2)$,
 $(1,2,1,0,0)(4,6,5,2,1,7,3)$, $(1,2,1,0,1)(5,6,2,4,1,7,3)$,
 $(1,2,1,0,2)(4,6,3,5,1,7,2)$, $(1,2,1,1,0)(5,6,2,1,3,7,4)$,
 $(1,2,1,1,1)(4,6,5,1,3,7,2)$, $(1,2,1,1,2)(5,6,2,1,4,7,3)$,
 $(1,2,1,2,0)(4,6,5,3,1,7,2)$, $(1,2,1,2,1)(4,6,5,3,2,7,1)$,
 $(1,2,1,2,2)(5,6,2,3,4,7,1)$, $(1,2,2,0,0)(5,6,3,2,1,7,4)$,
 $(1,2,2,0,1)(5,6,3,4,1,7,2)$, $(1,2,2,0,2)(5,6,3,2,4,7,1)$,
 $(1,2,2,1,0)(5,6,4,1,3,7,2)$, $(1,2,2,1,1)(5,6,3,4,2,7,1)$,
 $(1,2,2,1,2)(5,6,4,1,2,7,3)$, $(1,2,2,2,0)(5,6,4,2,3,7,1)$,
 $(1,2,2,2,1)(5,6,4,3,1,7,2)$, $(1,2,2,2,2)(5,6,4,3,2,7,1)$,
 $(2,0,0,0,0)(2,1,6,7,3,5,4)$, $(2,0,0,0,1)(1,5,6,7,3,4,2)$,
 $(2,0,0,0,2)(2,3,6,7,4,5,1)$, $(2,0,0,1,0)(2,4,6,7,3,5,1)$,
 $(2,0,0,1,1)(4,2,6,7,5,3,1)$, $(2,0,0,1,2)(3,1,6,7,2,4,5)$,
 $(2,0,0,2,0)(4,1,6,7,5,2,3)$, $(2,0,0,2,1)(4,1,6,7,3,2,5)$,
 $(2,0,0,2,2)(3,1,6,7,5,4,2)$, $(2,0,1,0,0)(3,2,6,7,4,1,5)$,
 $(2,0,1,0,1)(4,2,6,7,3,1,5)$, $(2,0,1,0,2)(3,2,6,7,1,4,5)$,
 $(2,0,1,1,0)(2,5,6,7,1,3,4)$, $(2,0,1,1,1)(4,5,6,7,2,3,1)$,
 $(2,0,1,1,2)(2,3,6,7,5,4,1)$, $(2,0,1,2,0)(4,5,6,7,1,2,3)$,
 $(2,0,1,2,1)(4,5,6,7,3,2,1)$, $(2,0,1,2,2)(3,1,6,7,4,2,5)$,
 $(2,0,2,0,0)(2,5,6,7,4,1,3)$, $(2,0,2,0,1)(5,3,6,7,2,1,4)$,
 $(2,0,2,0,2)(5,3,6,7,4,1,2)$, $(2,0,2,1,0)(3,4,6,7,2,1,5)$,

$(2,0,2,1,1)(3,4,6,7,2,5,1)$, $(2,0,2,1,2)(5,3,6,7,4,2,1)$,
 $(2,0,2,2,0)(3,4,6,7,5,1,2)$, $(2,0,2,2,1)(3,4,6,7,5,2,1)$,
 $(2,0,2,2,2)(5,1,6,7,4,2,3)$, $(2,1,0,0,0)(3,5,6,4,7,1,2)$,
 $(2,1,0,0,1)(4,1,6,5,7,3,2)$, $(2,1,0,0,2)(3,5,6,2,7,4,1)$,
 $(2,1,0,1,0)(4,1,6,2,7,5,3)$, $(2,1,0,1,1)(4,2,6,1,7,5,3)$,
 $(2,1,0,1,2)(4,3,6,1,7,5,2)$, $(2,1,0,2,0)(4,1,6,3,7,5,2)$,
 $(2,1,0,2,1)(4,2,6,3,7,5,1)$, $(2,1,0,2,2)(5,1,6,3,7,4,2)$,
 $(2,1,1,0,0)(4,5,6,2,7,1,3)$, $(2,1,1,0,1)(5,2,6,4,7,1,3)$,
 $(2,1,1,0,2)(4,3,6,5,7,1,2)$, $(2,1,1,1,0)(5,2,6,1,7,3,4)$,
 $(2,1,1,1,1)(4,5,6,1,7,3,2)$, $(2,1,1,1,2)(5,2,6,1,7,4,3)$,
 $(2,1,1,2,0)(4,5,6,3,7,1,2)$, $(2,1,1,2,1)(4,5,6,3,7,2,1)$,
 $(2,1,1,2,2)(5,2,6,3,7,4,1)$, $(2,1,2,0,0)(5,3,6,2,7,1,4)$,
 $(2,1,2,0,1)(5,3,6,4,7,1,2)$, $(2,1,2,0,2)(5,3,6,2,7,4,1)$,
 $(2,1,2,1,0)(5,4,6,1,7,3,2)$, $(2,1,2,1,1)(5,3,6,4,7,2,1)$,
 $(2,1,2,1,2)(5,4,6,1,7,2,3)$, $(2,1,2,2,0)(5,4,6,2,7,3,1)$,
 $(2,1,2,2,1)(5,4,6,3,7,1,2)$, $(2,1,2,2,2)(5,4,6,3,7,2,1)$,
 $(2,2,0,0,0)(3,5,6,4,1,7,2)$, $(2,2,0,0,1)(4,1,6,5,3,7,2)$,
 $(2,2,0,0,2)(3,5,6,2,4,7,1)$, $(2,2,0,1,0)(4,1,6,2,5,7,3)$,
 $(2,2,0,1,1)(4,2,6,1,5,7,3)$, $(2,2,0,1,2)(4,3,6,1,5,7,2)$,
 $(2,2,0,2,0)(4,1,6,3,5,7,2)$, $(2,2,0,2,1)(4,2,6,3,5,7,1)$,
 $(2,2,0,2,2)(5,1,6,3,4,7,2)$, $(2,2,1,0,0)(4,5,6,2,1,7,3)$,
 $(2,2,1,0,1)(5,2,6,4,1,7,3)$, $(2,2,1,0,2)(4,3,6,5,1,7,2)$,
 $(2,2,1,1,0)(5,2,6,1,3,7,4)$, $(2,2,1,1,1)(4,5,6,1,3,7,2)$,
 $(2,2,1,1,2)(5,2,6,1,4,7,3)$, $(2,2,1,2,0)(4,5,6,3,1,7,2)$,
 $(2,2,1,2,1)(4,5,6,3,2,7,1)$, $(2,2,1,2,2)(5,2,6,3,4,7,1)$,
 $(2,2,2,0,0)(5,3,6,2,1,7,4)$, $(2,2,2,0,1)(5,3,6,4,1,7,2)$,
 $(2,2,2,0,2)(5,3,6,2,4,7,1)$, $(2,2,2,1,0)(5,4,6,1,3,7,2)$,
 $(2,2,2,1,1)(5,3,6,4,2,7,1)$, $(2,2,2,1,2)(5,4,6,1,2,7,3)$,
 $(2,2,2,2,0)(5,4,6,2,3,7,1)$, $(2,2,2,2,1)(5,4,6,3,1,7,2)$,
 $(2,2,2,2,2)(5,4,6,3,2,7,1)$

Listing of the elements $\mathbf{x} \in Z_3^4$ and the corresponding values of $H(\mathbf{x}) \in S_6$.

$(0,0,0,0)(1,2,3,4,5,6)$, $(0,0,0,1)(1,2,3,6,4,5)$,
 $(0,0,0,2)(1,2,3,5,4,6)$, $(0,0,1,0)(1,4,2,6,5,3)$,
 $(0,0,1,1)(1,4,2,3,6,5)$, $(0,0,1,2)(1,4,2,5,6,3)$,
 $(0,0,2,0)(1,3,4,6,5,2)$, $(0,0,2,1)(1,3,4,5,6,2)$,
 $(0,0,2,2)(1,3,4,2,6,5)$, $(0,1,0,0)(1,5,3,4,6,2)$,
 $(0,1,0,1)(1,2,5,3,4,6)$, $(0,1,0,2)(1,5,3,2,4,6)$,

$(0,1,1,0)(1,5,2,4,6,3)$, $(0,1,1,1)(1,5,2,3,4,6)$,
 $(0,1,1,2)(1,4,5,2,6,3)$, $(0,1,2,0)(1,3,5,4,6,2)$,
 $(0,1,2,1)(1,5,4,3,6,2)$, $(0,1,2,2)(1,5,4,2,6,3)$,
 $(0,2,0,0)(1,6,3,4,5,2)$, $(0,2,0,1)(1,2,6,3,4,5)$,
 $(0,2,0,2)(1,6,3,2,4,5)$, $(0,2,1,0)(1,6,2,4,5,3)$,
 $(0,2,1,1)(1,6,2,3,4,5)$, $(0,2,1,2)(1,4,6,2,5,3)$,
 $(0,2,2,0)(1,3,6,4,5,2)$, $(0,2,2,1)(1,6,4,3,5,2)$,
 $(0,2,2,2)(1,6,4,2,5,3)$, $(1,0,0,0)(4,1,3,5,6,2)$,
 $(1,0,0,1)(4,1,3,6,5,2)$, $(1,0,0,2)(4,1,3,2,6,5)$,
 $(1,0,1,0)(3,1,2,4,6,5)$, $(1,0,1,1)(3,1,2,6,4,5)$,
 $(1,0,1,2)(3,1,2,5,4,6)$, $(1,0,2,0)(2,1,4,5,6,3)$,
 $(1,0,2,1)(2,1,4,3,6,5)$, $(1,0,2,2)(2,1,4,6,5,3)$,
 $(1,1,0,0)(6,1,3,4,5,2)$, $(1,1,0,1)(4,1,5,3,6,2)$,
 $(1,1,0,2)(6,1,3,2,4,5)$, $(1,1,1,0)(3,1,5,4,6,2)$,
 $(1,1,1,1)(6,1,5,3,4,2)$, $(1,1,1,2)(6,1,5,2,4,3)$,
 $(1,1,2,0)(2,1,5,4,6,3)$, $(1,1,2,1)(6,1,4,3,5,2)$,
 $(1,1,2,2)(6,1,4,2,5,3)$, $(1,2,0,0)(5,1,3,4,6,2)$,
 $(1,2,0,1)(4,1,6,3,5,2)$, $(1,2,0,2)(5,1,3,2,4,6)$,
 $(1,2,1,0)(5,1,2,4,6,3)$, $(1,2,1,1)(5,1,6,3,4,2)$,
 $(1,2,1,2)(5,1,6,2,4,3)$, $(1,2,2,0)(2,1,6,4,5,3)$,
 $(1,2,2,1)(5,1,4,3,6,2)$, $(1,2,2,2)(5,1,4,2,6,3)$,
 $(2,0,0,0)(4,2,1,5,6,3)$, $(2,0,0,1)(4,2,1,3,6,5)$,
 $(2,0,0,2)(4,2,1,6,5,3)$, $(2,0,1,0)(3,4,1,5,6,2)$,
 $(2,0,1,1)(3,4,1,6,5,2)$, $(2,0,1,2)(3,4,1,2,6,5)$,
 $(2,0,2,0)(2,3,1,4,6,5)$, $(2,0,2,1)(2,3,1,6,4,5)$,
 $(2,0,2,2)(2,3,1,5,4,6)$, $(2,1,0,0)(6,2,1,4,5,3)$,
 $(2,1,0,1)(6,2,1,3,4,5)$, $(2,1,0,2)(4,5,1,2,6,3)$,
 $(2,1,1,0)(3,5,1,4,6,2)$, $(2,1,1,1)(6,4,1,3,5,2)$,
 $(2,1,1,2)(6,5,1,2,4,3)$, $(2,1,2,0)(6,3,1,4,5,2)$,
 $(2,1,2,1)(2,5,1,3,4,6)$, $(2,1,2,2)(6,3,1,2,4,5)$,
 $(2,2,0,0)(5,2,1,4,6,3)$, $(2,2,0,1)(5,2,1,3,4,6)$,
 $(2,2,0,2)(4,6,1,2,5,3)$, $(2,2,1,0)(3,6,1,4,5,2)$,
 $(2,2,1,1)(5,4,1,3,6,2)$, $(2,2,1,2)(5,6,1,2,4,3)$,
 $(2,2,2,0)(5,3,1,4,6,2)$, $(2,2,2,1)(2,6,1,3,4,5)$,
 $(2,2,2,2)(5,3,1,2,4,6)$

Listing of the elements $\mathbf{x} \in Z_3^3$ and the corresponding values of $R(\mathbf{x}) \in S_5$.
 $(0,0,0)(1,2,3,5,4)$, $(0,0,1)(1,4,3,5,2)$, $(0,0,2)(1,5,3,4,2)$,

$(0,1,0)(1,2,4,5,3), (0,1,1)(1,4,2,5,3), (0,1,2)(1,5,4,3,2),$
 $(0,2,0)(1,2,5,4,3), (0,2,1)(1,4,5,3,2), (0,2,2)(1,3,5,4,2),$
 $(1,0,0)(4,1,3,5,2), (1,0,1)(5,1,3,4,2), (1,0,2)(2,1,3,5,4),$
 $(1,1,0)(3,1,4,5,2), (1,1,1)(5,1,4,3,2), (1,1,2)(2,1,4,5,3),$
 $(1,2,0)(4,1,5,3,2), (1,2,1)(5,1,2,4,3), (1,2,2)(2,1,5,4,3),$
 $(2,0,0)(4,2,1,5,3), (2,0,1)(5,4,1,3,2), (2,0,2)(2,5,1,4,3),$
 $(2,1,0)(3,2,1,5,4), (2,1,1)(3,4,1,5,2), (2,1,2)(3,5,1,4,2),$
 $(2,2,0)(4,3,1,5,2), (2,2,1)(5,3,1,4,2), (2,2,2)(2,3,1,5,4)$

Listing of the elements $\mathbf{x} \in Z_3^3$ and the corresponding values of $S(\mathbf{x}) \in S_5$.

$(0,0,0)(2,1,3,4,5), (0,0,1)(2,4,3,1,5), (0,0,2)(2,5,3,1,4),$
 $(0,1,0)(2,1,4,5,3), (0,1,1)(2,4,1,5,3), (0,1,2)(2,5,4,1,3),$
 $(0,2,0)(2,1,5,4,3), (0,2,1)(2,4,5,1,3), (0,2,2)(2,3,5,1,4),$
 $(1,0,0)(4,2,3,1,5), (1,0,1)(5,2,3,1,4), (1,0,2)(1,2,3,5,4),$
 $(1,1,0)(3,2,4,1,5), (1,1,1)(5,2,4,1,3), (1,1,2)(1,2,4,5,3),$
 $(1,2,0)(4,2,5,1,3), (1,2,1)(5,2,1,4,3), (1,2,2)(1,2,5,4,3),$
 $(2,0,0)(4,1,2,5,3), (2,0,1)(5,4,2,1,3), (2,0,2)(1,5,2,4,3),$
 $(2,1,0)(3,1,2,5,4), (2,1,1)(3,4,2,1,5), (2,1,2)(3,5,2,1,4),$
 $(2,2,0)(4,3,2,1,5), (2,2,1)(5,3,2,1,4), (2,2,2)(1,3,2,5,4)$

Listing of the elements $\mathbf{x} \in Z_3^4$ and the corresponding values of $T(\mathbf{x}) \in S_6$.

$(0,0,0,0)(2,4,3,1,5,6), (0,0,0,1)(2,4,3,6,1,5),$
 $(0,0,0,2)(2,4,3,5,1,6), (0,0,1,0)(2,1,4,6,5,3),$
 $(0,0,1,1)(2,1,4,3,6,5), (0,0,1,2)(2,1,4,5,6,3),$
 $(0,0,2,0)(2,3,1,6,5,4), (0,0,2,1)(2,3,1,5,6,4),$
 $(0,0,2,2)(2,3,1,4,6,5), (0,1,0,0)(2,5,3,1,6,4),$
 $(0,1,0,1)(2,4,5,3,1,6), (0,1,0,2)(2,5,3,4,1,6),$
 $(0,1,1,0)(2,5,4,1,6,3), (0,1,1,1)(2,5,4,3,1,6),$
 $(0,1,1,2)(2,1,5,4,6,3), (0,1,2,0)(2,3,5,1,6,4),$
 $(0,1,2,1)(2,5,1,3,6,4), (0,1,2,2)(2,5,1,4,6,3),$
 $(0,2,0,0)(2,6,3,1,5,4), (0,2,0,1)(2,4,6,3,1,5),$
 $(0,2,0,2)(2,6,3,4,1,5), (0,2,1,0)(2,6,4,1,5,3),$
 $(0,2,1,1)(2,6,4,3,1,5), (0,2,1,2)(2,1,6,4,5,3),$
 $(0,2,2,0)(2,3,6,1,5,4), (0,2,2,1)(2,6,1,3,5,4),$
 $(0,2,2,2)(2,6,1,4,5,3), (1,0,0,0)(1,2,3,5,6,4),$
 $(1,0,0,1)(1,2,3,6,5,4), (1,0,0,2)(1,2,3,4,6,5),$
 $(1,0,1,0)(3,2,4,1,6,5), (1,0,1,1)(3,2,4,6,1,5),$
 $(1,0,1,2)(3,2,4,5,1,6), (1,0,2,0)(4,2,1,5,6,3),$

$(1,0,2,1)(4,2,1,3,6,5)$, $(1,0,2,2)(4,2,1,6,5,3)$,
 $(1,1,0,0)(6,2,3,1,5,4)$, $(1,1,0,1)(1,2,5,3,6,4)$,
 $(1,1,0,2)(6,2,3,4,1,5)$, $(1,1,1,0)(3,2,5,1,6,4)$,
 $(1,1,1,1)(6,2,5,3,1,4)$, $(1,1,1,2)(6,2,5,4,1,3)$,
 $(1,1,2,0)(4,2,5,1,6,3)$, $(1,1,2,1)(6,2,1,3,5,4)$,
 $(1,1,2,2)(6,2,1,4,5,3)$, $(1,2,0,0)(5,2,3,1,6,4)$,
 $(1,2,0,1)(1,2,6,3,5,4)$, $(1,2,0,2)(5,2,3,4,1,6)$,
 $(1,2,1,0)(5,2,4,1,6,3)$, $(1,2,1,1)(5,2,6,3,1,4)$,
 $(1,2,1,2)(5,2,6,4,1,3)$, $(1,2,2,0)(4,2,6,1,5,3)$,
 $(1,2,2,1)(5,2,1,3,6,4)$, $(1,2,2,2)(5,2,1,4,6,3)$,
 $(2,0,0,0)(1,4,2,5,6,3)$, $(2,0,0,1)(1,4,2,3,6,5)$,
 $(2,0,0,2)(1,4,2,6,5,3)$, $(2,0,1,0)(3,1,2,5,6,4)$,
 $(2,0,1,1)(3,1,2,6,5,4)$, $(2,0,1,2)(3,1,2,4,6,5)$,
 $(2,0,2,0)(4,3,2,1,6,5)$, $(2,0,2,1)(4,3,2,6,1,5)$,
 $(2,0,2,2)(4,3,2,5,1,6)$, $(2,1,0,0)(6,4,2,1,5,3)$,
 $(2,1,0,1)(6,4,2,3,1,5)$, $(2,1,0,2)(1,5,2,4,6,3)$,
 $(2,1,1,0)(3,5,2,1,6,4)$, $(2,1,1,1)(6,1,2,3,5,4)$,
 $(2,1,1,2)(6,5,2,4,1,3)$, $(2,1,2,0)(6,3,2,1,5,4)$,
 $(2,1,2,1)(4,5,2,3,1,6)$, $(2,1,2,2)(6,3,2,4,1,5)$,
 $(2,2,0,0)(5,4,2,1,6,3)$, $(2,2,0,1)(5,4,2,3,1,6)$,
 $(2,2,0,2)(1,6,2,4,5,3)$, $(2,2,1,0)(3,6,2,1,5,4)$,
 $(2,2,1,1)(5,1,2,3,6,4)$, $(2,2,1,2)(5,6,2,4,1,3)$,
 $(2,2,2,0)(5,3,2,1,6,4)$, $(2,2,2,1)(4,6,2,3,1,5)$,
 $(2,2,2,2)(5,3,2,4,1,6)$